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# WEIGHTED JOIN-SEMILATTICES AND TRANSVERSAL MATROIDS

BY

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ABSTRACT. We investigate join-semilattices in which each element is assigned a nonnegative weight in a strictly increasing way. A join-subsemilattice of a Boolean lattice is weighted by cardinality, and we give a characterization of these in terms of the notion of a spread. The collection of flats with no coloops (isthmuses) of a matroid or pregeometry, partially ordered by set-theoretic inclusion, forms a join-semilattice which is weighted by rank. For transversal matroids these join-semilattices are isomorphic to join-subsemilattices of Boolean lattices. Using a previously obtained characterization of transversal matroids and results on weighted join-semilattices, we obtain another characterization of transversal matroids. The problem of constructing a transversal matroid whose join-semilattice of flats is isomorphic to a given join-subsemilattice of a Boolean lattice is then investigated.

1. **Introduction.** We are motivated by a recent study [3] of transversal matroids in which a characterization of transversal matroids is given in terms of the join-semilattice of flats with no coloops (isthmuses). The characterization depends on an algorithm used to construct a family of flats. We consider here the more general situation of a join-semilattice $J$ in which each element has assigned to it a nonnegative weight where the only assumption is that weight is a strictly increasing function on $J$. We introduce the concept of a spread for such a weighted join-semilattice, show a spread is unique if it exists at all, and give a characterization of join-semilattices of a finite Boolean lattice which are weighted by cardinality. Making use of the characterization of transversal matroids given in [3], we then give an alternate characterization which has the advantage of not depending on an algorithm. Finally we consider the problem of constructing a transversal matroid such that its weighted join-semilattice of flats with no coloops is isomorphic to a given join-subsemilattice of a Boolean lattice.

A family of objects differs from a set in that the objects may be repeated. We use parentheses ( ) to denote families and braces { } to denote sets. If $E$ is a

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set and $\mathfrak{A} = (a_i : i \in I)$ is a family of elements of $E$, then for $x \in E$ we define the *multiplicity* of $x$ in $\mathfrak{A}$ by

$$m(\mathfrak{A}, x) = |\{i \in I: a_i = x\}|.$$

This multiplicity may be 0. If $\mathfrak{A}_1, \mathfrak{A}_2$ are two families of elements of $E$, we write $\mathfrak{A}_1 \leq \mathfrak{A}_2$ provided $m(\mathfrak{A}_1, x) \leq m(\mathfrak{A}_2, x)$ for all $x \in E$. We consider two families to be identical if $m(\mathfrak{A}_1, x) = m(\mathfrak{A}_2, x)$ for all $x \in E$. The cardinality $\|\mathfrak{A}\|$ of a family $\mathfrak{A}$ is defined by $\|\mathfrak{A}\| = |I|$.

2. **Weighted join-semilattices.** A *join-semilattice* is a partially ordered set $J$ such that each pair $a$, $b$ of elements of $J$ has a least upper bound, which is denoted by $a \vee b$. A nonempty collection of subsets of a set $E$ which is closed under union is a join-semilattice; the partial order is set-theoretic inclusion. Such a join-semilattice is a join-subsemilattice of the Boolean lattice $B(E)$ of all subsets of $E$. A finite join-semilattice has a maximal element which we usually denote by 1; it need not have a minimal element but, if it does, it is usually denoted by 0. It is well known that a finite join-semilattice with a minimal element is a lattice [1].

If $P$ is a partially ordered set, then a mapping $\omega$ from $P$ to the nonnegative integers is a *weighting* of $P$ if $a < b$ implies $\omega(a) < \omega(b)$ $(a, b \in P)$. A weighting of $P$ need not be a grading [1] of $P$, for we do not assume that $\omega(b) = \omega(a) + 1$ if $b$ covers $a$. If $E$ is a set and $P$ is a collection of subsets of $E$ partially ordered by set-theoretic inclusion, then $\omega(A) = |A|$ defines a weighting of $P$. A partially ordered set with a specific weighting is called a *weighted partially ordered set.*

Let $P$ be a weighted partially ordered set with a maximal element 1 and set $\omega(1) = r$. Thus any chain has length at most $r$. A *spread* of $P$ is a family $\mathfrak{A} = (a_i : i \in I)$ of elements of $P$ such that the following condition holds: if $x \in P$ with $\omega(x) = r - k$ $(0 \leq k \leq r)$, then

$$|\{i \in I: a_i \geq x\}| = k. \tag{2.0.1}$$

In other words, exactly $k$ members of $\mathfrak{A}$ are greater than or equal to $x$ if $\omega(x) = r - k$. If $P$ has a minimal element 0, then clearly the cardinality $\|\mathfrak{A}\|$ of the spread $\mathfrak{A}$ equals $r - \omega(0)$. Thus if $\omega(0) = 0$, a spread of $P$ has exactly $r = \omega(1)$ elements. The notion of a spread arose in a characterization of transversal matroids [3], which will be fully explained in due course.

**Theorem 2.1.** *If the weighted partially ordered set P with maximal element has a spread, the spread is unique.*

**Proof.** Suppose $\mathfrak{A}$ is a spread of $P$, and let $\omega(1) = r$. Let $a \in P$ with $\omega(a) = r - k$. Then from (2.0.1) we conclude that the multiplicity of $a$ in $\mathfrak{A}$ is given by

(2.1.1) $$m(\mathfrak{A}, a) = k - \sum_{x>a} m(\mathfrak{A}, x).$$

This implies, in particular, that $\Sigma_{x>a} m(\mathfrak{A}, x) \le k$. The equation (2.1.1) along with $m(\mathfrak{A}, 1) = 0$ furnishes a recursion formula for the multiplicities of elements of $P$ in $\mathfrak{A}$. Since these multiplicities are uniquely determined, so is the spread $\mathfrak{A}$.

The proof of Theorem 2.1 gives an algorithm for determining a spread if one exists. If there is no spread, then there is some element $a$ of $P$ with $\omega(a) = r - k$ such that $\Sigma_{x>a} m(\mathfrak{A}, x) > k$.

The next theorem furnishes examples of weighted partially ordered sets which have spreads. Theorem 2.3 will then show that these examples are not very special.

**Theorem 2.2.** *Let $J$ be a join-subsemilattice of the Boolean lattice on a set $E$ with $|E| = r$. Suppose $E$ is the maximal element of $J$, and let $J$ be weighted by the cardinality function. Then*

(2.2.1) *$J$ has a spread $\mathfrak{A}$, and* $$\|\mathfrak{A}\| = r - \left|\bigcap \{A : A \in J\}\right|.$$

(2.2.2) $$m(\mathfrak{A}, A) = \left|\left(\bigcap X : A \subsetneq X \in J\right)\right| - |A| \qquad (A \in J).$$

**Proof.** The assumption that $E$ is the maximal element of $J$ is one of convenience. If this were not the case, $E$ would be replaced by a smaller set.

Let $A \in J$ with $|A| = r - k$. We need to show that $m(\mathfrak{A}, A) = k - \Sigma_{A \subsetneq X \in J} m(\mathfrak{A}, X)$ and $m(\mathfrak{A}, E) = 0$ allows us to define $\mathfrak{A}$ recursively. This is surely true for $k = 0$ (that is, $A = E$). We proceed by induction on $k$. Let $A \in J$ with $|A| = r - k$. Let $X_1, \cdots, X_t$ be the members of $J$ which strictly contain $A$. Thus $|X_i| = r - k_i$ where $k_i < k$ $(1 \le i \le t)$. Since $J$ is a join-semilattice, $X_{i_1} \cup \cdots \cup X_{i_s} \in J$ whenever $1 \le i_1 < \cdots < i_s \le t$; let $|X_{i_1} \cup \cdots \cup X_{i_s}| = r - k_{i_1 \cdots i_s}$ where $k_{i_1 \cdots i_s} < k$. Thus by induction the recursion has produced $k_{i_1 \cdots i_s}$ members of $\mathfrak{A}$ that contain $X_{i_1} \cup \cdots \cup X_{i_s}$. Hence by the principle of inclusion-exclusion exactly

$$n = \sum_{1 \le i \le t} k_i - \sum_{1 \le i_1 < i_2 \le t} k_{i_1 i_2} + \sum_{1 \le i_1 < i_2 < i_3 \le t} k_{i_1 i_2 i_3} - \cdots$$

members of $\mathfrak{A}$ that strictly contain $A$ have been produced. But

$$n = \sum_{1\le i\le t} |E\backslash X_i| - \sum_{1\le i_1<i_2\le t} |E\backslash(X_{i_1} \cup X_{i_2})|$$
$$+ \sum_{1\le i_1<i_2<i_3\le t} |E\backslash(X_{i_1} \cup X_{i_2} \cup X_{i_3})| - \cdots$$
$$= \left|\bigcup_{i=1}^{t} (E\backslash X_i)\right| = \left|E\backslash \bigcap_{i=1}^{t} X_i\right| = r - \left|\bigcap_{i=1}^{t} X_i\right|.$$

Since $A \subseteq X_i$ $(1 \le i \le t)$, $A \subseteq \bigcap_{i=1}^{t} X_i$ so that $|\bigcap_{i=1}^{t} X_i| \ge r - k$ and $n \le r - (r-k) = k$. Thus we can define $m(\mathfrak{A}, A)$ by $k - n \ge 0$. But $k - n = k - (r - |\bigcap_{i=1}^{t} X_i|) = |\bigcap_{i=1}^{t} X_i| - (r-k) = |\bigcap_{i=1}^{k} X_i| - |A|$. Thus we have proved that $J$ has a spread and that (2.2.2) is satisfied.

We have yet to prove that $\|\mathfrak{A}\| = r - |\bigcap\{A: A \in J\}|$. This is surely true if $\varnothing \in J$. If $\varnothing \notin J$, then $J^* = J \cup \{\varnothing\}$ is a join-subsemilattice of the Boolean lattice on $E$ and has a spread $\mathfrak{A}^*$ where $\|\mathfrak{A}^*\| = r$. But

$$\|\mathfrak{A}\| = \|\mathfrak{A}^*\| - m(\mathfrak{A}^*, \phi) = r - \left|\left(\bigcap X: X \in J\right)\right|$$

according to (2.2.2), and this establishes the formula for $\|\mathfrak{A}\|$.

Corollary 2.3. *If $\mathfrak{A} = (A_i: i \in I)$ is the spread of the join-subsemilattice $J$ of the Boolean lattice on $E$, then for $A \in J$,*

$$\left(\bigcap X: A \subsetneq X \in J\right) = \left(\bigcap A_i: A \subsetneq A_i,\ i \in I\right). \tag{2.3.1}$$

**Proof.** The set on the right side of (2.3.1) surely contains that on the left. Suppose now that $A \subsetneq X \in J$ but $X \ne A_i$ $(i \in I)$. Then arguing by induction $(|X| > |A|)$, and using (2.2.2) we conclude that

$$X = \left(\bigcap Y,\ X \subsetneq Y \in J\right) = \left(\bigcap A_i: X \subsetneq A_i,\ i \in I\right).$$

Thus if $A \subsetneq X \in J$, then either $X = A_i$ for some $i \in I$ or else there is $J \subseteq I$ such that $X = \bigcap_{i \in J} A_i$. Since $A \subsetneq X$, $A \subsetneq A_i$ $(i \in J)$ and this establishes (2.3.1).

Consider the join-semilattice $J$ of Theorem 2.2 and its spread $\mathfrak{A} = (A_i: i \in I)$. Let $A \in J$ and let $\mathfrak{A}_A$ be the subfamily of $\mathfrak{A}$ consisting of all members $A_i$ of $\mathfrak{A}$ with $A \subseteq A_i$. $\mathfrak{A}_A$ is the spread of the interval $[A, E]$ of $J$. Then for $A, B \in J$, $A \subseteq B$ if and only if $\mathfrak{A}_B \le \mathfrak{A}_A$. For if $A \subseteq B$, then surely $\mathfrak{A}_B \le \mathfrak{A}_A$. On the other hand if $\mathfrak{A}_B \le \mathfrak{A}_A$, then $B \in \mathfrak{A}_B$ implies $B \in \mathfrak{A}_A$ so that $A \subseteq B$, while $B \notin \mathfrak{A}_B$ implies $B \notin \mathfrak{A}$ (i.e. $m(\mathfrak{A}, B) = 0$), which by (2.2.2) and (2.3.1) implies

$$B = \bigcap(A_i: A_i \in \mathfrak{A}_B) \supseteq \bigcap(A_i: A_i \in \mathfrak{A}_A) \supseteq A.$$

Thus $A \subseteq B$. Hence the partial order of $J$ is determined by the partial order on the $\mathfrak{A}_A$ $(A \in J)$.

If $J_1$ and $J_2$ are two join-semilattices, an injection $\sigma: J_1 \to J_2$ is a *semilattice monomorphism* if $\sigma(a \vee b) = \sigma(a) \vee \sigma(b)$ for all $a, b \in J_1$. We shall be interested now in weighted semilattices which are isomorphic where the isomorphism preserves weights.

**Theorem 2.4.** *Let $J$ be a weighted join-semilattice with $\omega(1) = r$. Let $E$ be a set with $|E| = r$. Then there is a semilattice monomorphism $\sigma: J \to B(E)$, the Boolean lattice on $E$, such that $\omega(a) = |\sigma(a)|$ for all $a \in J$ if and only if $J$ has a spread of at most $r$ elements.*

**Proof.** By Theorem 2.2 if such a $\sigma$ exists, $J$ has a spread with at most $r$ elements. Suppose now $J$ has a spread $\mathfrak{A} = (a_i: i \in I)$ with $\|\mathfrak{A}\| = |I| \leq r$. For $a \in J$, let $I_a = \{i \in I: a_i \geq a\}$. Thus if $\omega(a) = r - k$, $|I_a| = k$. We define a map $\tau: J \to B(I)$ by $\tau(a) = I \setminus I_a$. Thus for $a, b \in J$, $\tau(a \vee b) = I \setminus I_{a \vee b}$. But $I_{a \vee b} = I_a \cap I_b$; for if $a_i \geq a \vee b$, then $a_i \geq a, b$ so that $I_{a \vee b} \subseteq I_a \cap I_b$ while if $a_i \in I_a \cap I_b$, then $a_i \geq a, b$ and thus $a_i \geq a \vee b$ so that $I_a \cap I_b \subseteq I_{a \vee b}$. This means that

$$\tau(a \vee b) = I \setminus (I_a \cap I_b) = (I \setminus I_a) \cup (I \setminus I_b) = \tau(a) \cup \tau(b).$$

Suppose that for some $a, b \in J$ with $a \neq b$, we have $\tau(a) = \tau(b)$. We may suppose that $b \not\leq a$ so that $a \vee b > a$. Then, by the above, $\tau(a \vee b) = \tau(a) \cup \tau(b) = \tau(a)$. Thus $I_{a \vee b} = I_a$. But since $\omega(a \vee b) > \omega(a)$, $|I_{a \vee b}| < |I_a|$, and we have a contradiction. Thus $\tau$ is a semilattice monomorphism from $J$ to $B(I)$. We calculate that for $a \in J$

$$|\tau(a)| = |I| - |I_a| = |I| - (r - \omega(a)) = \omega(a) - (r - |I|),$$

where $r - |I| \geq 0$. Let $t = r - |I|$ and let $I^*$ be a $t$ element set with $I \cap I^* = \emptyset$. Then $|I \cup I^*| = r$ and $\sigma: J \to B(I \cup I^*)$ defined by $\sigma(a) = \tau(a) \cup I^*$ is a semilattice isomorphism with $|\sigma(a)| = |\tau(a)| + t = \omega(a)$. This completes the proof of the theorem.

**3. Application to transversal matroids.** A characterization of transversal matroids is given by Brualdi and Dinolt [3]. We shall use the result of §2 to give an alternate formulation of it. But first we review briefly matroids, in general, and transversal matroids, in particular; for further details we refer the reader to [3] and the references contained within.

Let $E$ be a finite set. A *matroid* [5] M on $E$ (or *combinatorial pregeometry* [4]) is a nonempty collection of subsets of $E$, called *independent sets* such that

(i) a subset of an independent set is independent (thus $\emptyset \in \mathbf{M}$) (ii) $A_1, A_2 \in \mathbf{M}$ with $|A_1| < |A_2|$ imply $A_1 \cup \{x\} \in \mathbf{M}$ for some $x \in A_2 \setminus A_1$. Each subset $X$ of $E$ has a well-defined *rank* $\rho(X)$ which equals the common cardinality of all maximal independent sets contained in $X$. The rank of the matroid $\mathbf{M}$ equals $\rho(E)$. For $X \subseteq E$, $\mathbf{M}_X = \{A : A \in \mathbf{M}, A \subseteq X\}$ is a matroid, called the *restriction* of $M$ to $A$. A closure relation can be defined on the subsets of $E$ by defining $\overline{X}$ to be the largest subset of $E$ containing $X$ which has the same rank as $X$. Those subsets $F$ of $E$ with $\overline{F} = F$ are called *flats*. The collection $\mathfrak{L}(\mathbf{M})$ of flats of $\mathbf{M}$, partially ordered by set-theoretic inclusion, form a geometric lattice [4]. If $X \subseteq E$, then $x \in X$ is a *coloop* or *isthmus* of $X$ if $\rho(X \setminus \{x\}) = \rho(X) - 1$. The collection $\mathfrak{F}(\mathbf{M})$ of flats of $\mathbf{M}$ which have no coloops forms a join-subsemilattice of $\mathfrak{L}(\mathbf{M})$. Given a pair $F_1$, $F_2$ of flats in $\mathfrak{F}(\mathbf{M})$ with $F_1 \subseteq F_2$, such that no other flat of $\mathfrak{F}(\mathbf{M})$ lies between $F_1$ and $F_2$, then the interval $[F_1, F_2]$ of $\mathfrak{L}(\mathbf{M})$ consists of all sets of the form $F_1 \cup A$ where $A \subseteq F_2 \setminus F_1$, $|A| \leq \rho(F_2) - \rho(F_1) - 1$, along with $F_2$. Thus the flats of $\mathfrak{F}(\mathbf{M})$, given as subsets of $E$ with their rank, determine all flats of $\mathfrak{L}(\mathbf{M})$ as sets and thus the partial order of $\mathfrak{L}(\mathbf{M})$; that is, they determine the lattice $\mathfrak{L}(\mathbf{M})$.

A matroid $\mathbf{M}$ on $E$ is a *transversal matroid* provided there is a family $(A_1, \cdots, A_n)$ of subsets of $E$ such that $\mathbf{M} = \mathbf{M}(A_1, \cdots, A_n)$, the collection of partial transversals of $(A_1, \cdots, A_n)$. If $\mathbf{M}$ is a transversal matroid of rank $r$, then there are $r$ sets $A_1, \cdots, A_r$ such that $\mathbf{M} = \mathbf{M}(A_1, \cdots, A_r)$. The family $(A_1, \cdots, A_r)$ is called a *presentation* of $\mathbf{M}$. We recall Hall's theorem which says that the family $(A_1, \cdots, A_r)$ has a transversal (thus a system of distinct representatives) if and only if

$$\left| \bigcup_{i \in K} A_i \right| \geq |K| \qquad (K \subseteq \{1, \cdots, r\}).$$

Now let $\mathbf{M}$ be an arbitrary matroid of rank $r$ on the finite set $E$. We regard the join-semilattice $\mathfrak{F}(\mathbf{M})$ as a weighted join-semilattice by letting $\omega(F) = \rho(F)$ for $F \in \mathfrak{F}(\mathbf{M})$. The unique flat in $\mathfrak{F}(\mathbf{M})$ of weight 0 is the closure of the empty set. In the terminology of §2 the characterization of transversal matroids given in [3] is the following: $\mathbf{M}$ is a transversal matroid if and only if $\mathfrak{F}(\mathbf{M})$ has a spread $(F_1, \cdots, F_r)$ where $\rho(\bigcap_{i \in K} F_i) \leq r - |K|$ $(K \subseteq \{1, \cdots, r\})$. It is also proved in [3] that if $\mathbf{M}$ is a transversal matroid, then $\mathbf{M} = \mathbf{M}(E \setminus F_1, \cdots, E \setminus F_r)$; indeed $(E \setminus F_1, \cdots, E \setminus F_r)$ is the maximal presentation of $\mathbf{M}$. This means that we cannot enlarge any of the sets $E \setminus F_1, \cdots, E \setminus F_r$ and still have a presentation of $\mathbf{M}$. It is enough to know that the sets $F_1, \cdots, F_r$ have no coloops, in order to conclude that $(E \setminus F_1, \cdots, E \setminus F_r)$ is the maximal presentation of $\mathbf{M}$ ([1] and [3]).

If $G_1$, $G_2$ are flats in $\mathfrak{L}(M)$ with $G_1 \subseteq G_2$, then $\mathfrak{F}(M)_{[G_1,G_2]}$ denotes the join-subsemilattice of $\mathfrak{L}(M)$ consisting of all flats of $\mathfrak{L}(M)$ with no coloops which lie in the interval $[G_1, G_2]$ of $\mathfrak{L}(M)$. Note that $\mathfrak{F}(M) = \mathfrak{F}(M)_{[\emptyset,E]}$, and that $\mathfrak{F}(M)_{[G_1,G_2]}$ is a join-subsemilattice of $\mathfrak{F}(M)$. We regard $\mathfrak{F}(M)_{[G_1,G_2]}$ as weighted by rank (or we could assign $F \in \mathfrak{F}(M)_{[G_1,G_2]}$ the weight $\rho(F) - \rho(G_2)$).

**Theorem 3.1.** *Let* M *be a matroid of rank* $r$ *on the finite set* $E$. *Then* M *is a transversal matroid if and only if for all* $G_1, G_2 \in \mathfrak{L}(M)$ *with* $G_1 \subseteq G_2$ *the weighted join-semilattice* $\mathfrak{F}(M)_{[G_1,G_2]}$ *has a spread of at most* $\rho(G_2) - \rho(G_1)$ *members or, equivalently, there is a weight-preserving join-semilattice isomorphism from* $\mathfrak{F}(M)_{[G_1,G_2]}$ *to a Boolean lattice on an* $\rho(G_2) - \rho(G_1)$ *element set.*

**Proof.** By Theorem 2.4 the two criteria are equivalent. Suppose first that M is a transversal matroid of rank $r$. Then $\mathfrak{F}(M)$ has a spread $(F_1, \cdots, F_r)$. Let $G \in \mathfrak{L}(M)$ with $\rho(G) = r - k$. Then those members of $(F_1, \cdots, F_r)$ which contain $G$ are the members of a spread of $\mathfrak{F}(M)_{[G,E]}$. Let this spread be $(F_k: k \in K)$ where $K \subseteq \{1, \cdots, r\}$. Since $\rho(\bigcap_{i\in K} F_i) \leq r - |K|$ and since $G \subseteq \bigcap_{i\in K} F_i$ we have $\rho(G) = r - k \leq \rho(\bigcap_{i\in K} F_i)$, and we conclude that $|K| \leq k$. Thus $\mathfrak{F}(M)_{[G,E]}$ has a spread of at most $k$ members where $k = \rho(E) - \rho(G)$. Since $M_{G_2}$ is a transversal matroid of rank $\rho(G_2)$ on $G_2$ and $\mathfrak{F}(M)_{[G_1,G_2]}$ is isomorphic to $\mathfrak{F}(M_{G_2})_{[G_1,G_2]}$, we conclude that $\mathfrak{F}(M)_{[G_1,G_2]}$ has a spread of at most $\rho(G_2) - \rho(G_1)$ members for any $G_1, G_2 \in \mathfrak{L}(M)$ with $G_1 \subseteq G_2$.

Suppose now M is a matroid of rank $r$ such that for all $G \in \mathfrak{L}(M)$, $\mathfrak{F}(M)_{[G,E]}$ has a spread of at most $r - \rho(G)$ members. Thus, in particular, $\mathfrak{F}(M)$ has a spread $(F_1, \cdots, F_r)$ with $r$ members. Suppose for some $K \subseteq \{1, \cdots, r\}$, $\rho(\bigcap_{i\in K} F_i) > r - |K|$. Let $G = \bigcap_{i\in k} F_i$. Then $\mathfrak{F}(M)_{[G,E]}$ has a spread with at most $r - \rho(G)$ members. But a spread of $\mathfrak{F}(M)_{[G,E]}$ consists of all members of the spread of $\mathfrak{F}(M)$ which contain $G$; thus $F_i$ $(i \in K)$ are members of the spread of $\mathfrak{F}(M)_{[G,E]}$. We conclude that $|K| \leq r - \rho(G)$ or $\rho(G) \leq r - |K|$, and this is a contradiction. Hence $\rho(\bigcap_{i\in K} F_i) \leq r - |K|$ $(K \subseteq \{1, \cdots, r\})$ and M is a transversal matroid.

We mention one application to an interesting class of matroids. Let $E$ be a set and $\{X_i: 1 \leq i \leq k\}$ a collection of subsets of $E$ such that (i) $|X_i| \geq r - 1$ $(1 \leq i \leq k)$ and (ii) every $r - 1$ element subset of $E$ is a subset of exactly one of $X_1, \cdots, X_k$. Then [4] the set $E$, the $X_i$ $(1 \leq i \leq k)$, and all subsets $A$ of $E$ with $|A| \leq r - 2$ are the flats of a geometry (therefore matroid M) on $E$ of rank $r$. Such a geometry is called a Hartmanis geometry [4]. In this case $\mathfrak{F}(M)$ consists of those $X_i$ with $|X_i| \geq r$ (these are flats of rank $r - 1$), $\emptyset$, and possibly $E$. Thus $\mathfrak{F}(M)$ has a spread if and only if $|J| \leq r$ where $J = \{i: 1 \leq i \leq k, |X_i| \geq r\}$. The spread is then $(X_i: i \in J)$ along with the $\emptyset$ with the correct multiplicity to give $r$ sets in total.

**Theorem 3.2.** *The Hartmanis geometry* M *is a transversal geometry if and only if* $|\bigcap_{i\in I} X_i| \leq r - |I|$ $(I \subseteq J, |I| \geq 2)$.

*If* $I \subseteq J$, $|I| \geq 2$, *then* $\rho(\bigcap_{i\in I} X_i) = |\bigcap_{i\in I} X_i|$.

**4. Construction of transversal matroids.** Let M be a transversal matroid of rank $r$ on a finite set $E$, and let $\mathfrak{F}(M)$ be the join-semilattice of flats with no coloops, weighted by rank. Then we know there is a join-subsemilattice $J$ of the Boolean lattice on an $r$ element set such that $\mathfrak{F}(M)$ and $J$ are isomorphic as weighted join-semilattices. Since $\mathfrak{F}(M)$ has a minimal element $\bar{\phi}$, $\mathfrak{F}(M)$ is a lattice. (Note, however, $\mathfrak{F}(M)$ is not in general a sublattice of $\mathfrak{L}(M)$; it is, however, a join-subsemilattice of $\mathfrak{L}(M)$.) We consider the following question. Suppose $J$ is a join-subsemilattice with minimal element of the Boolean lattice on an $r$ element set, weighted by cardinality. Is there a transversal matroid M of rank $r$ such that $\mathfrak{F}(M)$ and $J$ are isomorphic as weighted join-semilattices?

**Theorem 4.1.** *Let $J$ be a join-subsemilattice of a Boolean lattice on an $r$ element set, weighted by cardinality, such that* $0, 1 \in J$ *with* $\omega(0) = 0$, $\omega(1) = r$. *Then there is a transversal matroid* M *of rank $r$ on a finite set $E$ such that* $\mathfrak{F}(M)$ *and $J$ are isomorphic as weighted partially ordered sets; that is, there is a bijection* $\sigma: J \rightarrow \mathfrak{F}(M)$ *such that*

(4.1.1) $$a < b \text{ if and only if } \sigma(a) < \sigma(b) \quad (a, b \in J),$$

(4.1.2) $$|a| = |\omega(\sigma(a))| \quad (a \in J).$$

We shall devote the remainder of this section to proving this theorem. The proof will be divided into several parts, but first we need a construction.

Let $E'$ be some sufficiently large set. Corresponding to each $a \in J$ we define a subset $F_a$ of $E'$ as follows:

(0) $F_0 = \emptyset$.

(1) If $a \in J$ with $\omega(a) = 1$, choose distinct elements $x, y$ of $E'$ and set $F_a = \{x, y\}$. We do this for each $a \in J$ of weight 1 in such a way that all elements chosen are distinct: $F_a \cap F_b = \emptyset$ if $a, b \in J$, $\omega(a) = \omega(b) = 1$, $a \neq b$.

⋮

($k$) If $a \in J$ with $\omega(a) = k$, let $J_a = \{x \in J: x < a\}$. For each $x \in J_a$, $\omega(x) < k$. If $a = \bigvee \{x: x \in J_a\}$, set $F_a = \bigcup \{F_x: x \in J_a\}$. If $\bigvee \{x \in J_a\} = b < a$ and $\omega(b) = l < k$, then choose a subset $X_a$ of $E'$ with $|X_a| = k - l + 1$ where the elements of $X_a$ are different from any chosen previously. Then set $F_a = F_b \cup X_a$. We do this for each element of $J$ of weight $k$ in such a way that $X_{a_1} \cap X_{a_2} = \varnothing$ whenever $\omega(a_1) = \omega(a_2) = k$ and $a_1 \neq a_2$.

⋮

The construction ends after we have gone through all elements of $J$. The family of sets $\mathfrak{F} = (F_a: a \in J)$ obtained is partially ordered by set-theoretic inclusion. Let $E = \bigcup_{a \in J} F_a$.

(4.1.3) $\qquad a \leq b$ if and only if $F_a \subseteq F_b \quad (a, b \in J)$.

Thus $a \neq b$ implies $F_a \neq F_b$, and the partially ordered sets $\mathfrak{F}$ and $J$ are isomorphic.

By construction it is clear that if $a \leq b$ then $F_a \subseteq F_b$. We need to prove conversely that $F_a \subseteq F_b$ implies $a \leq b$, and we do this by induction on weight. It is certainly true by construction if $a$ and $b$ have weight at most 1. Let $k > 1$ and assume that $F_a \subseteq F_b$ implies $a \leq b$ if $\omega(a) < k$, $\omega(b) < k$. Now consider $a$, $b \in J$ with $\omega(a) \leq k$, $\omega(b) \leq k$. We may assume by the induction that one of the latter is an equality.

We first make the following observation. For $x \in E$ let $\beta(x)$ be the element of $J$ such that $x \in X_{\beta(x)}$. Thus in the construction $x$ makes its first appearance in the set $F_{\beta(x)}$. It then follows for $x \in E$ and $c \in J$ that $\beta(x) \leq c$ if and only if $x \in F_c$.

Now if $a \neq \bigvee_{x < a} x$, then $X_a \neq \varnothing$. Let $z \in X_a$. Since $F_a \subseteq F_b$, $z \in F_b$; hence $a = \beta(z) \leq b$. Thus we may assume $a = \bigvee_{x < a} x$. Let $p = \bigvee_{z \in F_a} \beta(z)$. Thus $F_p = F_a$. Since $z \in F_a$ also implies $z \in F_b$, $\beta(z) \leq a, b$ for $z \in F_a$ and hence $p \leq a \wedge b$. If $p = a$, then $a \leq b$. If $p < a$, then consider $x \in J$ with $x < a$. We have $F_x \subseteq F_p$. Since $\omega(x)$, $\omega(p) < \omega(a) = k$, we have by induction that $x \leq p$. Hence $a = \bigvee_{x < a} x \leq p$. Since $p \leq a$, this implies $a = p$, a contradiction. Thus $a = p$ and $a \leq b$.

(4.1.4) The meet operation in the lattice $\mathfrak{F}$ is set-theoretic intersection.

Let $a, b \in J$ and $c = a \wedge b$, so that $F_c = F_a \wedge F_b$. Then $F_c \subseteq F_a \cap F_b$. Suppose there were an $x \in (F_a \cap F_b) \setminus F_c$; thus $\beta(x) \leq a, b$ so that $\beta(x) \leq a \wedge b = c$. This means $x \in F_c$, which is a contradiction.

We let the isomorphism $\sigma: J \to \mathfrak{F}$ where $\sigma(a) = F_a$ carry over the weight function of $J$ to $\mathfrak{F}$. That is, we define $\omega(F_a) = \omega(a)$ $(a \in J)$. Since $J$ is a join-subsemilattice of the Boolean lattice of an $r$ element set with 0 and 1, weighted by cardinality, $J$ has a spread $(a_i: 1 \leq i \leq r)$. Thus $(F_{a_i}: 1 \leq i \leq r)$ is the spread of $\mathfrak{F}$. Consider the transversal matroid $\mathbf{M} = \mathbf{M}(E \setminus F_{a_1}, \cdots, E \setminus F_{a_r})$. We have several things to verify concerning $\mathfrak{F}$ and the matroid $\mathbf{M}$.

(4.1.5) If $a > b$, then $|F_a \setminus F_b| \geq \omega(a) - \omega(b) + 1 \geq 2$.

To prove this we apply induction to $\omega(a)$. If $\omega(a) = 1$ then $\omega(b) = 0$ and by construction $F_b = \varnothing$, $|F_a| \geq 2$. Thus assume $\omega(a) = k$ and that the result holds when the weight is less than $k$. If there is $c \in J$ such that $a > c > b$, then by induction $|F_c \setminus F_b| \geq \omega(c) - \omega(b) + 1$. Thus if $|F_a \setminus F_c| \geq \omega(a) - \omega(c) + 1$ then $|F_a \setminus F_b| \geq \omega(a) - \omega(b) + 2$. Thus we might as well assume that there is no such $c$. If $x < a$ implies $x \leq b$, then $\bigvee \{x : x < a\} \leq b$. Thus by construction $|F_a \setminus F_b| = \omega(a) - \omega(b) + 1$. Otherwise there is an $x < a$ such that $x \not\leq b$. Then $\omega(x \wedge b) < \omega(x) < k$ so that by induction $A = F_x \setminus F_{x \wedge b} = F_x \setminus (F_x \cap F_b)$ has cardinality at least $\omega(x) - \omega(x \wedge b) + 1$. But since $J$ is a join-subsemilattice of a Boolean lattice and weighted by cardinality, $\omega(x \vee b) + \omega(x \wedge b) \leq \omega(x) + \omega(b)$. Since $b < x \vee b \leq a$, we have $a = x \vee b$. Thus $\omega(a) - \omega(b) \leq \omega(x) - \omega(x \wedge b)$. Since $A \subseteq F_a \setminus F_b$ and $|A| \geq \omega(a) - \omega(b) + 1$, we are done.

(4.1.6) The family $(E \setminus F_{a_1}, \cdots, E \setminus F_{a_r})$ has a transversal.

We need to show that the condition for the existence of a transversal is satisfied here. We calculate that for $\varnothing \neq K \subseteq \{1, \cdots, r\}$

$$\left| \bigcup_{i \in K} E \setminus F_{a_i} \right| = \left| E \setminus \bigcap_{i \in K} F_{a_i} \right| = |E| - |F_z|$$

where $z = \bigwedge \{a_i : i \in K\}$. But since $(a_i : 1 \leq i \leq r)$ is a spread of $J$, $\omega(\bigwedge_{i \in K} a_i) \leq r - |K|$; otherwise we contradict the definition of a spread. Thus $\omega(z) \leq r - |K|$. If we apply (4.1.5) with $a = 1$ $(F_1 = E)$ and $b = z$, we have

$$|E \setminus F_z| \geq r - \omega(z) + 1 \geq r - (r - |K|) + 1 = |K| + 1.$$

Thus we have a transversal.

(4.1.7) For $a \in J$, $F_a$ is a flat of **M**.

Let $\omega(a) = r - k$. Then exactly $k$ members of the spread $(a_i : 1 \leq i \leq r)$, say $a_1, \cdots, a_k$, satisfy $a_i \geq a$ $(i = 1, \cdots, k)$ and $a = \bigwedge (a_i : 1 \leq i \leq k)$. Since $\mathfrak{F}$ is lattice isomorphic to $J$ via $\sigma(a) = F_a$, $F_a = \bigcap (F_{a_i} : 1 \leq i \leq k)$. Thus

$$F_a \cap (E \setminus F_{a_i}) = \varnothing \qquad (1 \leq i \leq k).$$

Now let $x \in E \setminus F_a$. Thus $x \in \bigcup_{i=1}^{k} (E \setminus F_{a_i})$. If $B$ is a maximum partial transversal contained in $F_a$ (thus the rank of $F_a$ in **M** is $|B|$), then $B \cup x$ is also a partial transversal. Thus $F$ is closed and therefore a flat of **M**.

(4.1.8) If $a \in J$ has weight $r - k$, then the flat $F_a$ of $\mathbf{M}$ has rank equal to $r - k$. Thus the rank function coincides with the weight function on $\mathcal{F}$.

Since $\omega(a) = r - k$, there are exactly $k$ members of $(a_i: 1 \le i \le r)$, say $a_1, \cdots, a_k$, which are greater than or equal to $a$. Thus $F_a \subseteq F_{a_i}$ $(1 \le i \le k)$. Consider the family $((E \setminus F_{a_i}) \cap F_a: k+1 \le i \le r)$ of subsets of $F_a$. We show that this family has a transversal which will prove $\rho(F) = r - k$. Let $K \subseteq \{k+1, \cdots, r\}$. Then

$$\left| \bigcup_{i \in K} (E \setminus F_{a_i}) \cap F_a \right| = \left| \left( E \setminus \bigcap_{i \in K} F_{a_i} \right) \cap F_a \right| = \left| F_a \setminus \bigcap_{i \in K} \left( F_{a_i} \cap F_a \right) \right|.$$

Let $\bigcap_{i \in K} F_{a_i} \cap F_a = F_b$. Since $F_a = \bigcap_{i=1}^{k} F_{a_i}$, at least $k + |K|$ members of the spread $(F_{a_i}: 1 \le i \le r)$ of $\mathcal{F}$ contain $F_b$. Thus $\omega(F_b) \le r - (k + |K|)$, and so by (4.1.5)

$$|F_a \setminus F_b| \ge \omega(a) - \omega(b) + 1 \ge r - k - (r - (k + |K|)) + 1 = |K| + 1.$$

Thus the defined family has a transversal, which proves the statements made.

(4.1.9) For $a \in J$, the flat $F_a$ of $\mathbf{M}$ has no coloops.

We prove this by induction on $\rho(F_a) = \omega(F_a)$. If $\rho(F_a) = 0$ or $1$, this is true by construction. Let $\rho(F_a) = k > 1$, and assume the result is true for rank smaller than $k$. Suppose first $a = \bigvee_{x<a} x$. Then $F_a = \bigvee_{x<a} F_x$ and by induction each $F_x$ is a flat of $\mathbf{M}$ with no coloops. Since the join of flats with no coloops of a matroid has no coloops, this proves $F_a$ has no coloops. Suppose now $\bigvee_{x<a} x = b < a$ and thus $\rho(F_b) < k$. Since by induction $F_b$ has no coloops, no element of $F_b$ can be a coloop of $F_a$. But $F_a = F_b \cup X_a$ where $|X_a| = \rho(F_a) - \rho(F_b) + 1$. Thus if $B$ is a maximal independent set of $\mathbf{M}$ contained in $F_b$, then $B \cup (X_a \setminus x)$ is a maximal independent set contained in $F_a$. Thus no element of $X_a$ is a coloop of $F_a$, and $F_a$ has no coloops.

Finally we show

(4.1.10) If $F$ is a flat of $\mathbf{M}$ with no coloops, then $F = F_a$ for some $a \in J$.

Consider a flat $F$ of $\mathbf{M}$ with no coloops and let $\rho(F) = r - k$. Since $\mathbf{M} = \mathbf{M}(E \setminus F_{a_1}, \cdots, E \setminus F_{a_r})$ where $F_{a_i}$ is a flat with no coloops of $\mathbf{M}$ $(1 \le i \le r)$, then $(E \setminus F_{a_1}, \cdots, E \setminus F_{a_r})$ is the maximal presentation of $\mathbf{M}$. Thus $(F_{a_1}, \cdots, F_{a_r})$ is the spread of $\mathcal{F}(\mathbf{M})$ (recall it was defined to be the spread of $\mathcal{F}$). Thus since $\rho(F) = r - k$ there are exactly $k$ members of $(F_{a_1}, \cdots, F_{a_r})$, say $F_{a_1}, \cdots, F_{a_k}$

which contain $F$. Thus $F = \bigcap_{i=1}^{k} F_{a_i}$. But by (4.1.4) $\bigcap_{i=1}^{k} F_{a_i} = F_b$ for some $b \in J$. Thus $F = F_b$.

This now completes the proof of Theorem 4.1.

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